Towards Context-based Epistemic Logic

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Contents

Introduction

- **Purpose:**
- Context-based Epistemic Logic (CEL) = the logic of context-sensitive knowledge.
- (Traditional) Epistemic Logic (EL) = the logic of knowledge;

  – **CSK** is a notion abstracted from Epistemic Contextualism solution to Skepticism, which also being used to solve some other epistemic puzzles.

Introduction (cont'd)

- **Motivations**:
  - **1.** To provide a new approach to the study of **limited rationality** (LA) in Epistemic Logic.
  - Nowadays, the study of LA is extremely popular not only in the field of Epistemic Logic, but also in other fields like Game theory, Decision theory, Social software , and Artificial Intelligence (AI).

- As far as I know, there are mainly two approaches to deal with LA in EL:
  - Through awareness: (Cf. Fagin, 1988; & de Jager, 2009);
  - Through accessible: (Cf. Pacuit et al, 09).

– **2.** To strengthen further the connection between **epistemology** and **epistemic logic**.

– Quote:

"At first sight, the modern agenda of epistemology has little to do with logic… Now, epistemic logic started as a contribution to epistemology, or at least a tool in its modus operandi, with the seminal book ***Knowledge and Belief*** (Hintikka's, 1962,2005)."

---from (van Benthem, 2006)

Skepticism and EC

Skeptical Argument (SA)

- **Basic form** (DeRose, 1995):
  - P1: I don’t know that not-H.
  - P2: If I don’t know that not-H, then I don’t know that O.
  - C: So, I don’t know that O.

SA (cont'd)

- **Example:**
  - P1: I don't know that I am not a BIV.
  - P2: If I don't know that I am not a BIV, then I don't know that I have two hands.
  - C: So, I don't know that I have two hands.

Epistemic Contextualism (EC)

- **Quote:**

"…**EC** is the view that the proposition expressed by a given knowledge attribution (like "A knows that P" or "A knows that not P") depends upon the context in which it is uttered."

---from (Rysiew,2009).

EC's solution to Skepticism

- The presence of P1 has changed the context, such that a higher standard of knowledge are required.

- **Advantages:** explain the persuasiveness of SA & protect the correctness of our ordinary knowledge.

Further Question

- **Philosophers:** What is a context?
  - Cf. (Barke,2004); epistemic assumptions.
- **Logicians:** How to represent a context?
  - Cf. (Stalnaker,1998); a set of possible worlds (or states).

Epistemic Logic & Context Logic

Epistemic Logic (EL)

- **Language (L_E):**

$$\varphi ::= p | \neg\varphi | \varphi \wedge \psi | K\varphi;$$

Where $p \in P$.

Note: K is the abbreviation for K_a, since I only consider one agent *a* for simplicity.

EL (cont'd)

- **Epistemic model :**

$$M=<W,\approx,V>$$

Where:

– W is a non-empty set;

– $\approx$ is an equivalence relation on W;

– V is a valuation mapping each $p \in P$ to a subset of W, i.e., $V(p) \in 2^W$.

EL (cont'd)

- **Semantics:**

$M, w \models p$, iff $w \in V(p)$;

$M, w \models K\varphi$, iff for all $w' \in \approx_w$, $M, w' \models \varphi$;

where $\approx_w = \{v \mid v \in W \& w \approx v\}$.

EL (cont'd)

- **Axiomatization (S5):**
  - **Taut**: All instantiations of propositional tautologies;
  - **K**: $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$;
  - **T**: $K\varphi \rightarrow \varphi$; (Truth)
  - **4**: $K\varphi \rightarrow KK\varphi$; (Positive introspection)
  - **5**: $\neg K\varphi \rightarrow K\neg K\varphi$; (Negative introspection)
  - **MP**: From φ and $\varphi \rightarrow \psi$, infer ψ;
  - **N**: From φ, infer $K\varphi$.

EL (cont'd)

- **Notation:**
  - We denote EL-validity and EL-provability of φ as "$|=_{EL}\varphi$" and "$|-_{EL}\varphi$", respecitively.

- **Completeness of EL:**
  - **Theorem 1**: For any φ, $|=_{EL}\varphi$ iff $|-_{EL}\varphi$.
  - ***Proof.*** Cf. (van Ditmarsch et al, 2007, Chapter 7).

Context Logic (CL*)

- **Language (L_C):**

$$\varphi ::= p | \neg\varphi | \varphi \wedge \psi | [X]\varphi | [U]\varphi;$$

Where $p \in P$, and $X \in C$.

C is the index set of contexts, [X] is context operator, [U] is the universal modality.

Duals:

$\langle X\rangle\varphi = \neg[X]\neg\varphi$; $\langle U\rangle\varphi = \neg[U]\neg\varphi$.

CL (cont'd)

- **Context model :**

$$M=<W,R,V>$$

Where:

– W is a non-empty set;

– R is a function mapping each $X \in C$ to a subset of W, i.e., $R(X) \in 2^W$; *Henceforth we write R_X for R(X).

– V is a valuation mapping each $p \in P$ to a subset of W, i.e., $V(p) \in 2^W$.

CL (cont'd)

- **Semantics:**

M, w|=p, iff w$\in$V(p);

M, w|=[X]φ, iff for all w'$\in R_X$, M, w'|=φ;

M, w|=[U]φ, iff for all w'$\in$W, M, w'|=φ.

CL (cont'd)

- **Axiomatization ($\mathbf{K45^{XY}}$):**
  - **Taut** plus the following, where $X,Y \in C \cup \{U\}$:
  - $\mathbf{K^X}$: $[X](\varphi \rightarrow \psi) \rightarrow ([X]\varphi \rightarrow [X]\psi)$;
  - $\mathbf{T^U}$: $[U]\varphi \rightarrow \varphi$;
  - $\mathbf{4^{XY}}$:$[X]\varphi \rightarrow [Y][X]\varphi$;
  - $\mathbf{5^{XY}}$:$\langle X\rangle\varphi \rightarrow [Y]\langle X\rangle\varphi$;
  - **MP**: From φ and $\varphi \rightarrow \psi$, infer ψ;
  - $\mathbf{N^X}$: From φ, infer $[X]\varphi$.

CL (cont'd)

- **Notation:**
  - We denote CL-validity and CL-provability of φ as "$|=_{CL}\varphi$" and "$|-_{CL}\varphi$", respecitively.

- **Completeness of CL:**
  - **Theorem 2**: For any φ, $|=_{CL}\varphi$ iff $|-_{CL}\varphi$.
  - ***Proof.*** Cf. (Grossi et al, 2008).

Foundations of CEL

Foundations of CEL

- **Context-based epistemic model :**

$$M=<W,R,\approx,V>$$

Where:

– W is a non-empty set;

– R is a function mapping each $X \in C$ to a subset of W, i.e., $R_X \in 2^W$;

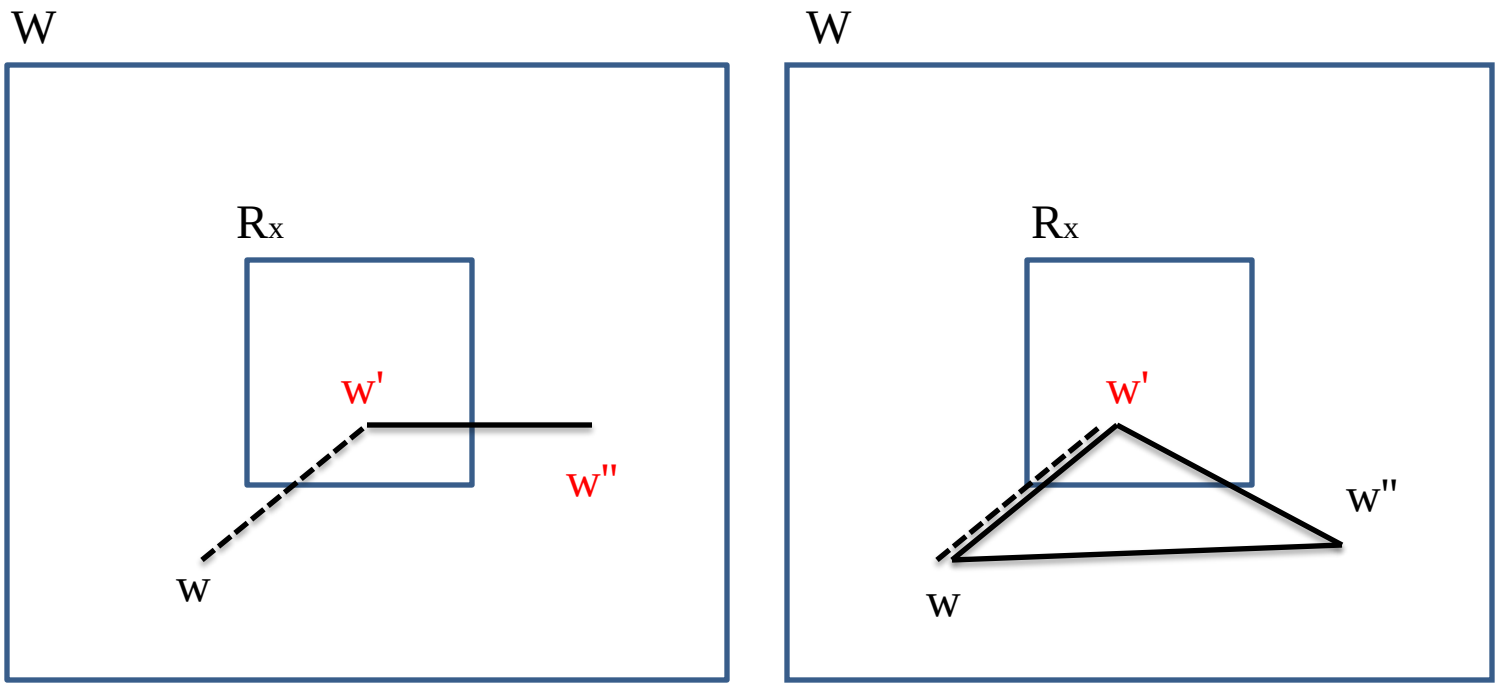
– $\approx$ is an equivalence relation on W;

– V is a valuation mapping each $p \in P$ to a subset of W, i.e., $V(p) \in 2^W$.

Foundations of CEL (cont'd)

- **Definitions of CSK:**
  - **Static style:**
    - M, w|=[X]Kφ, iff for all w' $\in R_X$, M, w'|=Kφ,
      iff for all w'' $\in \approx_{w'}$, M, w'|=φ.
  - **Dynamic style:**
    - M, w|=$K^{[X]}$φ, iff for all w' $\in \approx_w \cap R_X$, M, w'|=φ.
    - If we allow X to be U, then the standard epistemic operator K revives as $K^{[U]}$.

Contrasts



Static: w|=[X]Kφ, iff w'|=φ and w''|=φ.

Dynamic: w|= $K^{[X]}$φ, iff w'|=φ.

Figure 1. Definitions of CSK: Static vs. Dynamic

Contrasts (cont'd)

- **For static-style definition:**
  - (i) CSK is defined in term of context-free knowledge;* (which seems to be questionable.)
  - (ii) $K\varphi \rightarrow [X]K\varphi$ is not valid;
  - (iii) Nonetheless, $[X]K\varphi \rightarrow [X]\varphi$ is valid.
- **For dynamic-style definition:**
  - (i) CSK is defined independently;
  - (ii) $K\varphi \rightarrow K^{[X]}\varphi$ is valid;
  - (iii) Nonetheless, $K^{[X]}\varphi \rightarrow [X]\varphi$ is not valid.

Contrasts (cont'd)

– **Static-style** seems to be consistent with an **objective** understanding of context, since $[X]K\varphi \rightarrow [X]\varphi$ is valid. (*However, I haven't discovered any concrete example yet.)

– **Dynamic-style** seems to work well with the **subjective** understanding of context (esp. as **common assumptions**). Example: (Cf. next page)

SA in Dynamic CSK

- **For example:**

  p: I am not a BIV;

  q: I feel that I have two hands;

  r: I have to hands.

  The model is indicated as **Figure 2** on the right side, where V(p)={u, v}; V(q)={u, s}; V(r)={u}.

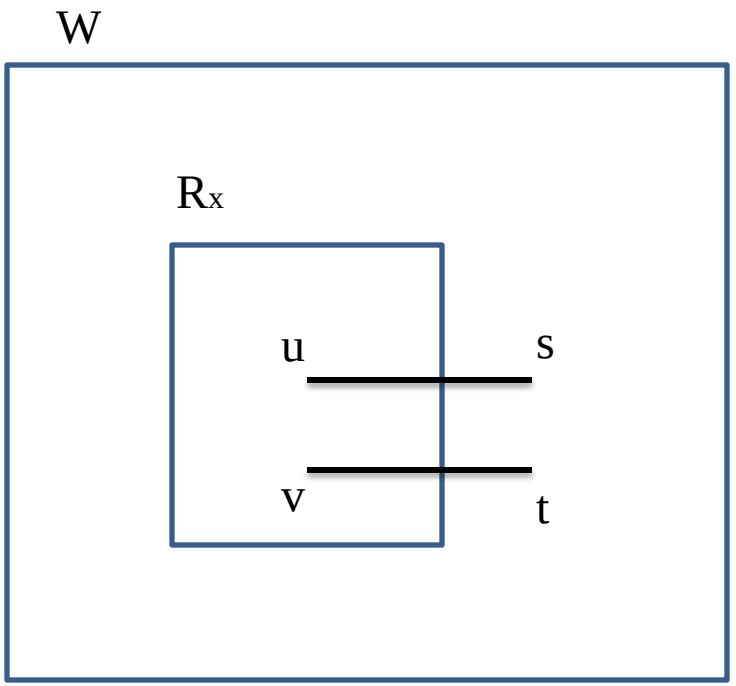


Figure 2. SA in Dynamic CSK

SA in Dynamic CSK (cont'd)

(i) In all states, I know my feelings (i.e., either I feel that I have two hands or not).

(ii) If w is the real state, then with the assumption of X, I know that I have two hands and I am not a BIV.

(iii) When P1 of SA appears, the context has been extended to contain either s or t (depending upon which of u and v take as the actual state).

(iv) So, the extension of context corresponds to the retraction of assumption. *(In next paper, I will revisit this example with more details after I work out the whole dynamic mechanics of DCEL.)

Candidates: SCEL & DCEL

SCEL

- **Language (L_{SCE}):**

$$\varphi ::= p | \neg\varphi | \varphi \wedge \psi | K\varphi | [X]\varphi | [U]\varphi;$$

Where $p \in P$, and $X \in C$.

- **Semantics:**

$M, w \models K\varphi$, iff for all $w' \in \approx_w$, $M, w' \models \varphi$;

$M, w \models [X]\varphi$, iff for all $w' \in R_X$, $M, w' \models \varphi$;

$M, w \models [U]\varphi$, iff for all $w' \in W$, $M, w' \models \varphi$.

SCEL (cont'd)

- **Axiomatization:**
  - All axioms and rules of EL and CL, plus the axiom schemas below:
  - $\mathbf{4^{XK}}$: $[X]\varphi \rightarrow K[X]\varphi$;
  - $\mathbf{5^{XK}}$: $\langle X\rangle\varphi \rightarrow K\langle X\rangle\varphi$.

***Remark:** Knowledge operator is semi-context, since the following schemas are generally invalid:

- $\mathbf{4^{KY}}$: $K\varphi \rightarrow [Y]K\varphi$;
- $\mathbf{5^{KY}}$: $\langle K\rangle\varphi \rightarrow [Y]\langle K\rangle\varphi$.

SCEL (cont'd)

- **Notation:**
  - We denote SCEL-validity and SCEL-provability of φ as "$|=_{SCEL}\varphi$" and "$|-_{SCEL}\varphi$", respecitively.

- **Completeness of CL:**
  - **Theorem 3**: For any φ, $|=_{SCEL}\varphi$ iff $|-_{SCEL}\varphi$.
  - ***Proof.*** Cf. (Xu, forthcoming).

DCEL

- **Language (L_{DCE}):**

$$\varphi ::= p|\neg\varphi|\varphi\wedge\psi|[X]\varphi|[U]\varphi|K^{[X]}\varphi|K^{[U]}\varphi;$$

Where $p\in P$, and $X\in C$.

- **Semantics:**

$M, w|=[X]\varphi$, iff for all $w'\in R_X$, $M, w'|=\varphi$;

$M, w|=[U]\varphi$, iff for all $w'\in W$, $M, w'|=\varphi$;

$M, w|=K^{[X]}\varphi$, iff for all $w'\in\approx_w\cap R_X$, $M, w'|=\varphi$;

$M, w|=K^{[U]}\varphi$, iff for all $w'\in\approx_w\cap W$, $M, w'|=\varphi$.

Notice that, $K^{[U]}$ is the same as K of SCEL, so DCEL is an extension of SCEL, and henceforth we write $K^{[U]}$ as K.

DCEL (cont'd)

- **Axiomatization:**
  - All axioms and rules of SCEL, plus the axiom schemas and rule below, where $X \in C$:
  - $\mathbf{K^{K[X]}}$: $K^{[X]}(\varphi \rightarrow \psi) \rightarrow (K^{[X]}\varphi \rightarrow K^{[X]}\psi)$;
  - $K\varphi \vee [X]\varphi \rightarrow K^{[X]}\varphi$;
  - $\mathbf{N^{K[X]}}$: From φ, infer $K^{[X]}\varphi$.

DCEL (cont'd)

- **Notation:**
  - We denote DCEL-validity and DCEL-provability of φ as "$|=_{DCEL}\varphi$" and "$|-_{DCEL}\varphi$", respecitively.

- **Completeness of DCEL:**
  - **Theorem 3**: For any φ, $|=_{DCEL}\varphi$ iff $|-_{DCEL}\varphi$.
  - ***Proof.*** Cf. (Xu, forthcoming).

Conclusion

- I have introduced the philosophical background of CSK and preliminaries of CEL: EL & CL;
- After that, I have proposed two distinct ways of defining CSK and made some detailed contrast;
- Further, I have obtained two candidate systems of CEL (namely, SCEL & DCEL) and proved their completeness.

Future Work

- **Future work:**
  - Develop the dynamic version of SCEL and DCEL;
    - (E.g., DCL; Cf. Aucher et al, 2009).
  - Compare with update semantics;
    - (Cf., Veltman, 1996; & de Jager, 2009).
  - Extend with more philosophical discussion;
    - (Cf. Lewis, 1996).
  - …

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Thank you!